Change of Basis

1. Coordinate Vector Relative to Basis:

Let $B = (b_1, b_2, \ldots b_n)$ be an ordered basis for a vector space V. If v is a vector in V, then $v = r_1 b_1 + r_2 b_2 + \ldots r_n b_n$. The coordinate vector of v relative to B, denoted by v_B, is $v_B = [r_1, r_2, \ldots r_n]$.

2. Change of Basis Matrix:

For this section, we need V_B to be a column vector. I.e. $V_B = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}$ instead of $[r_1, r_2, \ldots r_n]$.

Let M_B be the matrix having the vectors in the ordered basis B as column vectors. This is the basis matrix for B. B is an ordered basis for R^n.

$$M_B = \begin{bmatrix} | & | & & | \\ b_1 & b_2 & \ldots & b_n \\ | & | & & | \end{bmatrix}$$

Note: $M_B V_B = V$, where v is a vector in R^n

Furthermore, a vector space can have more than 1 basis.

So, if B' is another ordered basis for R^n, then $M_{B'} V_{B'} = V$.

This shows that $M_B V_B = M_{B'} V_{B'}$ and
$V_{B'} = M^{-1}_{B'} M_B V_B$.

Therefore, for any 2 ordered bases B and B',
there exists an invertible matrix C, s.t.
$C = M^{-1}_{B'} M_B$. C is called the change of basis
matrix.

Furthermore, for all v in $\mathbb{R}^n$,
$V_{B'} = C V_B$

Important Notations:
$C_{B,B'}$ means we are changing from coordinates
relative to B to coordinates relative to B'.
I.e. you read the subscripts from left to right.

Inverse of Change of Coordinate Matrix:
Suppose we have $V_{B'} = C_{B,B'} V_B$. Then,
$V_B = C^{-1}_{B,B'} V_{B'}$, where $C^{-1}_{B,B'}$ is the inverse
of $C_{B,B'}$. Note: $C_{B',B} = C^{-1}_{B,B'}$.

Steps For Finding the Change of Coordinate Matrix:

We know $C = M^{-1}_{B'} M_B$. However, if $M^{-1}_{B'}$ is
not already available, we need to form
the augmented matrix $[M_{B'} | M_B]$ and then
RREF it to $[I | C]$.

Another way of thinking about this is

$$\left[\begin{array}{cccc|cccc} | & | & & | & | & | & & | \\ b_1' & b_2' & \ldots & b_n' & b_1 & b_2 & \ldots & b_n \\ | & | & & | & | & | & & | \end{array}\right]$$

New basis ($M_{B'}$) Old basis (M_B)

$\sim [I | C_{B,B'}]$

E.g. 1 Let $B = ([1,1,0], [2,0,1], [1,-1,0])$ and let $E = (e_1, e_2, e_3)$ be the standard ordered basis of $\mathbb{R}^3$. Find the change of coordinate matrix $C_{E,B}$.

Solution

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

New Basis (M_B) Old Basis (M_E)

$$\sim \left[\begin{array}{ccc|ccc} 0 & 2 & 2 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & -1 \end{array}\right]$$

I $C_{E,B}$

$$\therefore C_{E,B} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -1 \\ 0 & 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} & -1 \end{bmatrix}$$